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Certain new classes of Multivalent Functions of order Alpha and Type Beta

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Abstract: In this paper, a new class δ - $TS_p(n,q,\alpha,\beta)$ consisting of δ - uniformly starlike, analytic, q-valent functions with negative coefficients of order α and type β and an another new class δ - $TUC(n,q,\alpha,\beta)$ consisting of δ-uniformly convex, analytic, q-valent functions with negative coefficients of order α and type β are introduced.

Coefficient estimates, distortion theorems, closure theorems and convolution results for functions in these classes are obtained.

AMS Subject Classification: 30C45

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1 Introduction :

Let $T(n,q)$ denote the class of functions of the form

$$f(z) = z^q - \sum_{k=n}^{\infty} a_{k+q} z^{k+q} \quad (a_{k+q} \geq 0; q, n \in \mathbb{N}) \tag{1}$$

which are analytic in the open unit disk $U = \{z \in \mathbb{C}; |z| < 1\}$.

Let us now define two new subclasses δ-$TS_p(n,q,\alpha,\beta)$ and δ-$TUC(n,q,\alpha,\beta)$ of $T(n,q)$.

Definition 1. let δ -$TS_p(n,q,\alpha,\beta)$ denote the class of δ-uniformly starlike, analytic, q-valent functions of order α and type β, satisfying the inequality

$$Re\left\{\frac{zf'(z)}{f(z)} - \alpha\right\} > \delta\left|\frac{zf'(z)}{f(z)} - \beta\right|, \tag{2}$$

$(0 \leq \alpha < \beta \leq q; 0 \leq \delta < q; z \in U)$.

Definition 2. let δ-$TUC(n,q,\alpha,\beta)$ denote the class of δ-uniformly convex, analytic, q-valent functions of order α and type β, satisfying the inequality

$$Re\left\{1+\frac{zf''(z)}{f'(z)}-\alpha\right\}>\delta\left|1+\frac{zf''(z)}{f'(z)}-\beta\right|, \tag{3}$$

$(0\le\alpha<\beta\le q; 0\le\delta<q; z\in U)$.

It follows from (2) and (3) that

$f\in\delta$-$TUC(n,q,\alpha,\beta)\Leftrightarrow zf'\in\delta$-$TS_p(n,q,\alpha,\beta)$.

Specializing the parameters n,q,α,β and δ, we obtain the following subclasses studied by other authors.

(i) $1-TS_p(1,1,\alpha,1) = 1-ST(\alpha,1) = S_p(\alpha) = S_pT(\alpha) = T^*\left(\frac{1+\alpha}{2}\right)$
and
$1-TUC(1,1,\alpha,1)=1-UCV(\alpha,1)=UCV(\alpha)=UCT(\alpha)=C^*\left(\frac{1+\alpha}{2}\right)$
(see [1], [2], [3], [8]);

(ii) $1-TS_p(1,1,0,1)=1-ST(0,1)=S_p$
and
$1-TUC(1,1,0,1)=1-UCV(0,1)=UCV$ (see [3] - [8]).

The object of this paper is to study various interesting properties of functions belonging to the classes δ-$TS_p(n,q,\alpha,\beta)$ and δ-$TUC(n,q,\alpha,\beta)$.

2 Coefficient Estimates :

Theorem 3. *A function of the form (1) is in the class* δ-$TS_p(n,q,\alpha,\beta)$ *if and only if*

$$\sum_{k=n}^{\infty}[(k+q-\alpha)+\delta(k+q-\beta)]a_{k+q}<(q-\alpha)+\delta(q-\beta) \tag{4}$$

$(0\le\alpha<\beta\le q; n,q\in\mathbb{N}; a_{k+q}\ge 0; 0\le\delta<q; z\in U)$.

Proof. Let $f\in\delta$-$TS_p(n,q,\alpha,\beta)$.

Then

$$Re\left\{\frac{zf'(z)}{f(z)}-\alpha\right\}-\delta\left|\frac{zf'(z)}{f(z)}-\beta\right|>0$$

$$Re\left\{\frac{qz^{q-1}-\sum_{k=n}^{\infty}(k+q)a_{k+q}z^{k+q-1}}{z^{q-1}-\sum_{k=n}^{\infty}a_{k+q}z^{k+q-1}}-\alpha\right\}$$

$$-\delta\left|\frac{qz^{q-1}-\sum_{k=n}^{\infty}(k+q)a_{k+q}z^{k+q-1}}{z^{q-1}-\sum_{k=n}^{\infty}a_{k+q}z^{k+q-1}}-\beta\right|>0$$

$$Re\left\{(q-\alpha)z^{q-1}-\sum_{k=n}^{\infty}(k+q-\alpha)a_{k+q}z^{k+q-1}\right\}$$

$$-\delta\left|(q-\beta)z^{q-1}-\sum_{k=n}^{\infty}(k+q-\beta)a_{k+q}z^{k+q-1}\right|>0$$

Letting z to take real values and $|z|\to 1$, we get

$$(q-\alpha)+\delta(q-\beta)-\sum_{k=n}^{\infty}[(k+q-\alpha)+\delta(k+q-\beta)]a_{k+q}>0$$

This implies

$$\sum_{k=n}^{\infty}[(k+q-\alpha)+\delta(k+q-\beta)]a_{k+q}<(q-\alpha)+\delta(q-\beta),$$

Conversely,

$$Re\left\{\frac{zf'(z)}{f(z)}-\alpha\right\}-\delta\left|\frac{zf'(z)}{f(z)}-\beta\right|$$

$$\geq 1-\left|\frac{zf'(z)}{f(z)}-\alpha-1\right|-\delta\left|\frac{zf'(z)}{f(z)}-\beta\right|$$

$$=1-\left|\frac{qz^{q-1}-\sum_{k=n}^{\infty}(k+q)a_{k+q}z^{k+q-1}}{z^{q-1}-\sum_{k=n}^{\infty}a_{k+q}z^{k+q-1}}-\alpha-1\right|$$

$$- \delta \left| \frac{qz^{q-1} - \sum_{k=n}^{\infty}(k+q)a_{k+q}z^{k+q-1}}{z^{q-1} - \sum_{k=n}^{\infty} a_{k+q}z^{k+q-1}} - \beta \right|$$

$$= 1 - \left| \frac{(q-\alpha)z^{q-1} - \sum_{k=n}^{\infty}(k+q-\alpha)a_{k+q}z^{k+q-1}}{z^{q-1} - \sum_{k=n}^{\infty} a_{k+q}z^{k+q-1}} - 1 \right|$$

$$- \delta \left| \frac{(q-\beta)z^{q-1} - \sum_{k=n}^{\infty}(k+q-\beta)a_{k+q}z^{k+q-1}}{z^{q-1} - \sum_{k=n}^{\infty} a_{k+q}z^{k+q-1}} \right|$$

$$\geq 1 + \frac{(q-\alpha)|z^{q-1}| - \left|\sum_{k=n}^{\infty}(k+q-\alpha)a_{k+q}z^{k+q-1}\right|}{\left|z^{q-1} - \sum_{k=n}^{\infty} a_{k+q}z^{k+q-1}\right|}$$

$$- 1 + \frac{\delta(q-\beta)|z^{q-1}| - \left|\sum_{k=n}^{\infty}\delta(k+q-\beta)a_{k+q}z^{k+q-1}\right|}{\left|z^{q-1} - \sum_{k=n}^{\infty} a_{k+q}z^{k+q-1}\right|}$$

$$= \frac{(q-\alpha) + \delta(q-\beta) - \sum_{k=n}^{\infty}[(k+q-\alpha) + \delta(k+q-\beta)]a_{k+q}}{\left|z^{q-1} - \sum_{k=n}^{\infty} a_{k+q}z^{k+q-1}\right|}$$

$$> 0 \quad \text{by (4)}.$$

Thus $f \in \delta\text{-}TS_p(n,q,\alpha,\beta)$. □

Corollary 4. *If $f \in \delta\text{-}TS_p(n,q,\alpha,\beta)$, then*

$$a_{k+q} \leq \frac{(q-\alpha) + \delta(q-\beta)}{(k+q-\alpha) + \delta(k+q-\beta)}, \tag{5}$$

$(0 \le \alpha < \beta \le q; 0 \le \delta < q; k \ge n; q, n \in \mathbb{N}).$

The equality holds for the function

$$f(z) = z^q - \frac{(q-\alpha) + \delta(q-\beta)}{(k+q-\alpha) + \delta(k+q-\beta)} z^{k+q}, \tag{6}$$

$(0 \le \alpha < \beta \le q; 0 \le \delta < q; k \ge n; q, n \in \mathbb{N}; z \in U).$

Using the same technique used in Theorem 3, we get the following theorem.

Theorem 5. *A function of the form (1) is in the class* δ- $TUC(n, q, \alpha, \beta)$ *if and only if*

$$\sum_{k=n}^{\infty}(k+q)[(k+q-\alpha) + \delta(k+q-\beta)]a_{k+q} < q[(q-\alpha) + \delta(q-\beta)], \tag{7}$$

$(0 \le \alpha < \beta \le q; 0 \le \delta < q; q, n \in \mathbb{N}; a_{k+q} \ge 0).$

Corollary 6. *If* $f \in \delta$-$TUC(n, q, \alpha, \beta)$ *then*

$$a_{k+q} \le \frac{q[(q-\alpha) + \delta(q-\beta)]}{(k+q)[(k+q-\alpha) + \delta(k+q-\beta)]}, \tag{8}$$

$(0 \le \alpha < \beta \le q; 0 \le \delta < q; k \ge 0; q, n \in \mathbb{N}).$

The equality holds for the function

$$f(z) = z^q - \frac{q[(q-\alpha) + \delta(q-\beta)]}{(k+q)[(k+q-\alpha) + \delta(k+q-\beta)]} z^{k+q} \tag{9}$$

$(0 \le \alpha < \beta \le q; 0 \le \delta < q; k \ge 0; q, n \in \mathbb{N}; z \in U).$

3 Distortion Theorems :

Theorem 7. *If the function* $f \in \delta$-$TS_p(n, q, \alpha, \beta)$ *then*

$$|z|^q - \frac{(q-\alpha)+\delta\,(q-\beta)}{(q+1-\alpha)+\delta\,(q+1-\beta)}|z|^{q+1} \le |f(z)|$$

$$\le |z|^q + \frac{(q-\alpha)+\delta\,(q-\beta)}{(q+1-\alpha)+\delta\,(q+1-\beta)}|z|^{q+1}$$

The result is sharp.

Proof. In view of Theorem 3, since $\psi(k+q) = (k+q-\beta) + \delta(k+q-\beta)$ is an increasing function of $k (k \ge n)$,we have

$$\psi(q+1)\sum_{k=n}^{\infty}|a_{k+q}| \leq \sum_{k=n}^{\infty}\psi(k+q)|a_{k+q}| \leq (q-\alpha)+\delta(q-\beta)$$

Therefore
$$\sum_{k=n}^{\infty}|a_{k+q}| \leq \frac{(q-\alpha)+\delta(q-\beta)}{\psi(q+1)}.$$

Thus we have
$$|f(z)| \leq |z|^q + \sum_{k=n}^{\infty}|a_{k+q}||z|^{q+1}$$

$$|f(z)| \leq |z|^q + \frac{(q-\alpha)+\delta\,(q-\beta)}{(q+1-\alpha)+\delta\,(q+1-\beta)}|z|^{q+1}$$

Similarly we get,
$$|f(z)| \geq |z|^q - \sum_{k=n}^{\infty}|a_{k+q}||z|^{q+1}$$

$$|f(z)| \geq |z|^q - \frac{(q-\alpha)+\delta\,(q-\beta)}{(q+1-\alpha)+\delta\,(q+1-\beta)}|z|^{q+1}$$

This completes the proof of Theorem 7.

Finally the result is sharp for the following function
$$f(z) = z^q - \frac{(q-\alpha)+\delta\,(q-\beta)}{(q+1-\alpha)+\delta\,(q+1-\beta)}z^{q+1} \tag{10}$$

$$(0 \leq \alpha < \beta \leq q; 0 \leq \delta < q; q, n \in \mathbb{N}; z \in U).$$ □

Theorem 8. *If the function $f \in \delta\text{-}TS_p(n,q,\alpha,\beta)$, then*
$$|z|^{q-1}\left[q - \frac{(q+1)[(q-\alpha)+\delta(q-\beta)]}{(q+1-\alpha)+\delta(q+1-\beta)}|z|\right] \leq |f'(z)|$$
$$\leq |z|^{q-1}\left[q + \frac{(q+1)[(q-\alpha)+\delta(q-\beta)]}{(q+1-\alpha)+\delta(q+1-\beta)}|z|\right]$$
The result is sharp.

Proof. Similarly, $\frac{\psi(k+q)}{k+q}$ is an increasing function of k, $(k \geq n)$.
In view of Theorem 3, we have
$$\frac{\psi(q+1)}{(q+1)}\sum_{k=n}^{\infty}(k+q)|a_{k+q}| \leq \sum_{k=n}^{\infty}\psi(k+q)|a_{k+q}| \leq (q-\alpha)+\delta(q-\beta)$$

Therefore,

$$\sum_{k=n}^{\infty}(k+q)|a_{k+q}| \leq \frac{(q+1)[(q-\alpha)+\delta(q-\beta)]}{\psi(q+1)}.$$

Thus we have,

$$|f'(z)| \leq |z|^{q-1}\left[q+\sum_{k=n}^{\infty}(k+q)|a_{k+q}||z|\right]$$

$$\leq |z|^{q-1}\left[q+\frac{(q+1)[(q-\alpha)+\delta(q-\beta)]}{(q+1-\alpha)+\delta(q+1-\beta)}|z|\right].$$

Similarly

$$|f'(z)| \geq |z|^{q-1}\left[q-\sum_{k=n}^{\infty}(k+q)|a_{k+q}||z|\right]$$

$$\geq |z|^{q-1}\left[q-\frac{(q+1)[(q-\alpha)+\delta(q-\beta)]}{(q+1-\alpha)+\delta(q+1-\beta)}|z|\right].$$

Finally the result is sharp for the function $f(z)$ given by (10). □

Using the same technique used in the Theorem 7 and Theorem 8, we get the following theorems.

Theorem 9. *If the function $f \in \delta\text{-}TUC(n,q,\alpha,\beta)$, then*

$$|z|^q - \frac{(q-\alpha)+\delta(q-\beta)}{(q+1)(q+1-\alpha)+\delta(q+1-\beta)}|z|^{q+1} \quad \leq \quad |f(z)|$$

$$\leq |z|^q + \frac{(q-\alpha)+\delta(q-\beta)}{(q+1)[(q+1-\alpha)+\delta(q+1-\beta)]}|z|^{q+1}.$$

The result is sharp for the function

$$f(z) = z^q - \frac{(q-\alpha)+\delta(q-\beta)}{(q+1)\ (q+1-\alpha)+\delta(q+1-\beta)}\ z^{\ q+1}, \tag{11}$$

$(0 \leq \alpha < \beta \leq q; 0 \leq \delta < q; q \in \mathbb{N}; z \in U)$.

Theorem 10. *If the function $f \in \delta\text{-}TUC(n,q,\alpha,\beta)$, then*

$$|z|^{q-1}\left[q-\frac{(q-\alpha)+\delta(q-\beta)}{[(q+1-\alpha)+\delta(q+1-\beta)}|z|\right] \quad \leq \quad |f'(z)|$$

$$\leq |z|^{q-1}\left[q+\frac{(q-\alpha)+\delta(q-\beta)}{[(q+1-\alpha)+\delta(q+1-\beta)]}|z|\right].$$

The result is sharp for the function given by (11).

4 Closure Theorems :

Theorem 11. *If* $f_q(z) = z^q$ *and*

$$f_{k+q}(z) = z^q - \frac{(q-\alpha)+\delta(q-\beta)}{(k+q-\alpha)+\delta(k+q-\beta)}z^{k+q},$$

then $f \in \delta\text{-}TS_p(n,q,\alpha,\beta)$ *if and only if it can be expressed in the form*

$$f(z) = \sum_{k=0}^{\infty}\lambda_{k+q}f_{k+q}(z),\ \text{where } \lambda_{k+q}\geq 0 \text{ and } \sum_{k=0}^{\infty}\lambda_{k+q} = 1,\ (q\in\mathbb{N}).$$

Proof. Let $f(z) = \sum_{k=0}^{\infty}\lambda_{k+q}f_{k+q}(z)$, where $\lambda_{k+q} \geq 0$ and $\sum_{k=1}^{\infty}\lambda_{k+q} = 1,\ (q\in\mathbb{N})$.

We have

$$f(z) = \sum_{k=0}^{\infty}\lambda_{k+q}f_{k+q}(z) = \lambda_q z^q + \sum_{k=1}^{\infty}\lambda_{k+q}\left[z^q - \frac{(q-\alpha)+\delta(q-\beta)}{(k+q-\alpha)+\delta(k+q-\beta)}z^{k+q}\right]$$

$$= \sum_{k=0}^{\infty}\lambda_{k+q}z^q - \sum_{k=1}^{\infty}\lambda_{k+q}\frac{(q-\alpha)+\delta(q-\beta)}{(k+q-\alpha)+\delta(k+q-\beta)}z^{k+q}$$

$$= z^q - \sum_{k=1}^{\infty}\lambda_{k+q}\frac{(q-\alpha)+\delta(q-\beta)}{(k+q-\alpha)+\delta(k+q-\beta)}z^{k+q}$$

Since

$$\sum_{k=1}^{\infty}[(k+q-\alpha)+\delta(k+q-\beta)]\lambda_{k+q}\frac{(q-\alpha)+\delta(q-\beta)}{(k+q-\alpha)+\delta(k+q-\beta)}$$

$$= [(q-\alpha)+\delta(q-\beta)]\sum_{k=1}^{\infty}\lambda_{k+q}$$

$$= [(q-\alpha)+\delta(q-\beta)](1-\lambda_q) < (q-\alpha)+\delta(q-\beta)$$

The condition (4) for $f(z) = \sum_{k=0}^{\infty}\lambda_{k+q}f_{k+q}(z)$ is satisfied.

Thus $f \in \delta\text{-}TS_p(n,q,\alpha,\beta)$.

Conversely, we suppose that $f \in \delta\text{-}TS_p(n,q,\alpha,\beta)$,

$$f(z) = z^q - \sum_{k=n}^{\infty} a_{k+q} z^{k+q}, \ a_{k+q} \geq 0$$

and we take

$$\lambda_{k+q} = \frac{(k+q-\alpha) + \delta(k+q-\beta)}{(q-\alpha)+\delta(q-\beta)} a_{k+q} \geq 0,$$

$(0 \leq \alpha < \beta \leq q; 0 \leq \delta < q; k \geq q; q \in \mathbb{N})$ with $\lambda_q = 1 - \sum_{k=n}^{\infty} \lambda_{k+q}$.

Using the condition (4), we obtain

$$\sum_{k=n}^{\infty} \lambda_{k+q} = \frac{1}{[(q-\alpha)+\delta(q-\beta)]} \sum_{k=n}^{\infty} [(k+q-\alpha)+\delta(k+q-\beta)] a_{k+q}$$
$$< \frac{1}{[(q-\alpha)+\delta(q-\beta)]} [(q-\alpha)+\delta(q-\beta)] = 1$$

so that $1 - \lambda_q < 1$ or $\lambda_q > 0$.

This completes our proof. □

Corollary 12. *The extreme points of δ-$TS_p(n, q, \alpha, \beta)$ are $f_q(z) = z^q$ and*

$$f_{k+q}(z) = z^q - \frac{(q-\alpha)+\delta(q-\beta)}{(k+q-\alpha)+\delta(k+q-\beta)} z^{k+q}, \ (k \geq n; q, n \in \mathbb{N}).$$

Using the same technique used in Theorem 11, we get the following theorem.

Theorem 13. *If $f_q(z) = z^q$ and*

$$f_{k+q}(z) = z^q - \frac{(q-\alpha)+\delta(q-\beta)}{(k+q)[(k+q-\alpha)+\delta(k+q-\beta)]} z^{k+q},$$

then $f \in \delta$-$TUC(n, q, \alpha, \beta)$ if and only if it can be expressed in the form

$f(z) = \sum_{k=0}^{\infty} \lambda_{k+q} f_{k+q}(z)$, *where* $\lambda_{k+q} \geq 0$ *and* $\sum_{k=0}^{\infty} \lambda_{k+q} = 1$, $(q \in \mathbb{N})$.

Corollary 14. *The extreme points of δ-$TUC(n, q, \alpha, \beta)$ are $f_q(z) = z^q$ and*

$$f_{k+q}(z) = z^q - \frac{(q-\alpha)+\delta(q-\beta)}{(k+q)[(k+q-\alpha)+\delta(k+q-\beta)]} z^{k+q},$$

$(k \geq n; q, n \in \mathbb{N})$.

5 Hadamard Product :

Definition 15. For the two functions $f, g \in T(n, q)$, with

$$f(z) = z^q - \sum_{k=n}^{\infty} a_{k+q} z^{k+q} \quad (a_{k+q} \geq 0; q, n \in \mathbb{N})$$

and

$$g(z) = z^q - \sum_{k=n}^{\infty} b_{k+q} z^{k+q} \quad (b_{k+q} \geq 0; q, n \in \mathbb{N}),$$

the modified Hadamard product (or convolution) $f * g$ is defined by

$$(f * g)(z) = z^q - \sum_{k=n}^{\infty} a_{k+q} b_{k+q} z^{k+q}.$$

Theorem 16. *If $f, g \in \delta\text{-}TS_p(n, q, \alpha, \beta)$ with*

$$f(z) = z^q - \sum_{k=n}^{\infty} a_{k+q} z^{k+q} \quad (a_{k+q} \geq 0; n, q \in \mathbb{N})$$

and

$$g(z) = z^q - \sum_{k=n}^{\infty} b_{k+q} z^{k+q} \quad (b_{k+q} \geq 0; n, q \in \mathbb{N}),$$

Then $f * g \in \delta\text{-}TS_p(n, q, \alpha, \beta)$

Proof. We have

$$\sum_{k=n}^{\infty} [(k + q - \alpha) + \delta(k + q - \beta)] a_{k+q} < (q - \alpha) + \delta(q - \beta)$$

and

$$\sum_{k=n}^{\infty} [(k + q - \alpha) + \delta(k + q - \beta)] b_{k+q} < (q - \alpha) + \delta(q - \beta)$$

We notice that $(f * g)(z) = z^q - \sum_{k=n}^{\infty} a_{k+q} b_{k+q} z^{k+q}$.

From $g \in T(n, q)$, by using theorem, we have $\sum_{k=n}^{\infty} (k + q) b_{k+q} \leq 1$

and we notice that $b_{k+q} < 1$, $(k + q = q + 1, q + 2, \ldots ; q \in \mathbb{N})$.

Thus,

$$\sum_{k=n}^{\infty} [(k + q - \alpha) + \delta(k + q - \beta)] a_{k+q} b_{k+q}$$

$$< \sum_{k=n}^{\infty} [(k+q-\alpha) + \delta(k+q-\beta)] a_{k+q} < (q-\alpha) + \delta(q-\beta)$$

Hence $f * g \in \delta\text{-}TS_p(n, q, \alpha, \beta)$. □

Using the same technique used in Theorem 16, we get Theorem 17.

Theorem 17. *If $f, g \in \delta\text{-}TUC(n, q, \alpha, \beta)$ with*

$$f(z) = z^q - \sum_{k=n}^{\infty} a_{k+q} z^{k+q} \quad (a_{k+q} \geq 0; n, q \in \mathbb{N})$$

and

$$g(z) = z^q - \sum_{k=n}^{\infty} b_{k+q} z^{k+q} \quad (b_{k+q} \geq 0; n, q \in \mathbb{N}),$$

then $f * g \in \delta\text{-}TUC(n, q, \alpha, \beta)$.

References :

[1] H. Silverman, Univalent functions with negative coecients, Proc. Amer. Maths Soc., 51(1) (1975), 109-116.

[2] R. Bharati, R. Parvatham and A. Swaminathan, On subclasses of uniformly convex functions and corresponding class of starlike functions, Tamkang Journal of Mathematics, 28(1) (1996), 17-32.

[3] F. Ronning, Uniformly convex functions and a corresponding class of starlike functions, Proc. Amer. Math Soc., 118(1) (1993), 189-196.

[4] A.W. Goodman, On uniformly convex functions, Annales Polonici Mathematici, 56(1) (1991), 87-92.

[5] A.W. Goodman, On uniformly starlike functions, Journal of Mathematical Analysis and Applications, 155(2) (1991), 364-370.

[6] F. Ronning, On starlike functions associated with parabolic regions, Annales Universitatis Mariae Curie-sklodowska, Section A, 45 (1991), 117-122.

[7] W.C. Ma and D. Minda, Uniformly convex functions, Annales Polonici Mathamatici, 57(2) (1992), 165-175.

[8] R.M. El-Ashwah, M.K. Aouf, A.A.M. Hassan and A.H. Hassan, Certain new classes of analytic functions with varying arguments, Journal of Complex Analysis, 2013, Article Id 958210, 1-5.